whose positive roots are $x = 1$, or $x = \dfrac{3 \pm \sqrt{5}}{2}$.

If $x = 1$, then $D = \dfrac{2}{\sqrt{2}} = \sqrt{2}$.

If $x = \dfrac{3+\sqrt{5}}{2}$, then $D = \dfrac{2}{\sqrt{3 - \left(\dfrac{3+\sqrt{5}}{2}\right)}} = \sqrt{6 + 2\sqrt{5}}$.

If $x = \dfrac{3-\sqrt{5}}{2}$, then $D = \sqrt{6 - 2\sqrt{5}}$.

Finally, note that if $x = 1$, then the two trapezoids are the same unit square.

Solution 2 by Kee-Wai Lau, Hong Kong, China

Call the first trapezoid $ABCD$ such that $AB = BC = CD = 1$ and $DA = x$.

Let $\angle ABC = \theta$ so that $\angle ADC = \pi - \theta$. Applying the cosine formula to triangles ABC and ADC, we obtain respectively

$$AC^2 = 2(1 - \cos\theta) \text{ and } AC^2 = x^2 + 2x\cos\theta + 1.$$

Eliminating AC from these two equations, we obtain $\cos\theta = \dfrac{1-x}{2}$ and hence $AC = \sqrt{x+1}$.

Let the diameter of the circle be d. By the sine formula, we have

$$d = \frac{AC}{\sin\theta} = \frac{2}{\sqrt{3-x}}. \qquad (1)$$

Call the second trapezoid $PQRS$ such $PQ = QR = RS = x$ and $SP = 1$.

Let $\angle$PQR=ϕ so that $\angle PSR = \pi - \phi$. By the procedure similar to that for trapezoid $ABCD$, we obtain

$$d = \frac{PR}{\sin\phi} = \frac{2x\sqrt{x}}{\sqrt{3x-1}}. \qquad (2)$$

From (1) and (2), we obtain $x^4 - 3x^3 + 3x - 1 = 0$, whose positive roots are $1,\ \dfrac{3-\sqrt{5}}{2},\ \dfrac{3+\sqrt{5}}{2}$. The corresponding values of d are $\sqrt{2},\ \sqrt{5}-1$, and $\sqrt{5}+1$.

Also solved by Paul M. Harms, North Newton, KS; Enkel Hysnelaj, University of Technology, Sydney, Australia (jointly with) Elton Bojaxhiu, Kriftel, Germany; Charles McCracken, Dayton, OH; John Nord, Spokane, WA; Boris Rays, Brooklyn, NY; Raul A. Simon, Santiago, Chile; Trey Smith, San Angelo, TX; Jim Wilson, Athens, GA, and the proposer.

- **5154:** *Proposed by Andrei Răzvan Băleanu (student, George Cosbuc National College) Motru, Romania*

Let a, b, c be the sides, m_a, m_b, m_c the lengths of the medians, r the in-radius, and R the circum-radius of the triangle ABC. Prove that:

$$\frac{m_a^2}{1+\cos A} + \frac{m_b^2}{1+\cos B} + \frac{m_c^2}{1+\cos C} \geq 6Rr\left(\frac{a}{b+c} + \frac{b}{c+a} + \frac{c}{a+b}\right).$$

Solution by Arkady Alt, San Jose, California, USA

Since

$$\begin{aligned}\frac{m_a^2}{1+\cos A} &= \frac{m_a^2}{2\cos^2\frac{A}{2}} = \frac{m_a^2}{2}\left(1+\tan^2\frac{A}{2}\right) = \frac{m_a^2}{2}+\frac{m_a^2}{2}\tan^2\frac{A}{2} \\ &= \frac{m_a^2}{2}+\frac{m_a^2}{2}\cdot\frac{r^2}{(s-a)^2} = \frac{2\left(b^2+c^2\right)-a^2}{8}+\frac{(b+c)^2-a^2+(b-c)^2}{8}\cdot\frac{r^2}{(s-a)^2} \\ &\geq \frac{2\left(b^2+c^2\right)-a^2}{8}+\frac{s(s-a)r^2}{2(s-a)^2} = \frac{2\left(b^2+c^2\right)-a^2}{8}+\frac{sr^2}{2(s-a)},\end{aligned}$$

then

$$\sum_{cyc}\frac{m_a^2}{1+\cos A} \geq \frac{3\left(a^2+b^2+c^2\right)}{8}+\frac{sr^2}{2}\sum_{cyc}\frac{1}{s-a}.$$

Noting that

$$\begin{aligned}\frac{sr^2}{2}\sum_{cyc}\frac{1}{s-a} &= \frac{(s-a)(s-b)(s-c)}{2}\sum_{cyc}\frac{1}{s-a} = \frac{1}{2}\sum_{cyc}(s-b)(s-c) \\ &= \frac{2ab+2bc+2ca-a^2-b^2-c^2}{8},\end{aligned}$$

we obtain

$$\frac{3\left(a^2+b^2+c^2\right)}{8}+\frac{sr^2}{2}\sum_{cyc}\frac{1}{s-a} = \frac{ab+bc+ca+a^2+b^2+c^2}{4}.$$

Hence,

$$\sum_{cyc}\frac{m_a^2}{1+\cos A} \geq A, \text{ where } \mathrm{A} = \frac{ab+bc+ca+a^2+b^2+c^2}{4}.$$

Also since,

$$6Rr = \frac{12Rrs}{2s} = \frac{3abc}{2s} \quad \text{and} \quad \frac{a}{\mathrm{b}+\mathrm{c}} \leq \frac{a^2(b+c)}{4abc},$$

we have,

$$B \geq 6Rr\sum_{cyc}\frac{a}{b+c}, \text{ where } \mathrm{B} = \frac{3abc}{2s}\sum_{\mathrm{cyc}}\frac{a^2(b+c)}{4abc} = \frac{3}{4(a+b+c)}\sum_{\mathrm{cyc}}a^2(b+c).$$

Thus, it suffices to prove inequality $A \geq B$.

Since $\sum_{cyc} a(a-b)(a-c) \geq 0$ (by the Schur Inequality), we have

$$4(a+b+c)(A-B) = (a+b+c)\left(ab+bc+ca+a^2+b^2+c^2\right)-3\sum_{cyc}a^2(b+c)$$

$$
\begin{aligned}
&= && (a+b+c)\left((a+b+c)^2-ab-bc-ca\right)\\
&- && 3(a+b+c)(ab+bc+ca)+9abc\\
&\Longleftrightarrow && 9abc+(a+b+c)^3 \ge 4(a+b+c)(ab+bc+ca)\\
&\Longleftrightarrow && \sum_{cyc} a(a-b)(a-c) \ge 0.
\end{aligned}
$$

Also solved by Kee-Wai Lau, Hong Kong, China, and the proposer.

- **5155:** *Proposed by José Luis Díaz-Barrero, Barcelona, Spain*

Let a, b, c, d be the roots of the equation $x^4+6x^3+7x^2+6x+1=0$. Find the value of

$$\frac{3-2a}{1+a}+\frac{3-2b}{1+b}+\frac{3-2c}{1+c}+\frac{3-2d}{1+d}.$$

Solution 1 by Valmir Bucaj (student, Texas Lutheran University), Seguin, TX

Since a, b, c, d are the roots of the equation $x^4+6x^3+7x^2+6x+1=0$, by Viéte's formulas we have

$$
\begin{aligned}
a+b+c+d &= -6\\
ab+ac+ad+bc+bd+cd &= 7\\
abc+abd+acd+bcd &= -6\\
abcd &= 1.
\end{aligned}
$$

For convenience we adopt the following notation:

$$
\begin{aligned}
x &= a+b+c+d\\
y &= ab+ac+ad+bc+bd+cd\\
z &= abc+abd+acd+bcd\\
w &= abcd.
\end{aligned}
$$

Finally, we have:

$$
\begin{aligned}
\frac{3-2a}{1+a}+\frac{3-2b}{1+b}+\frac{3-2c}{1+c}+\frac{3-2d}{1+d} &= \frac{-(8w+3z-2y-7x-12)}{w+z+y+x+1}\\
&= \frac{-(8\times 1+3\times(-6)-2\times 7-7\times(-6)-12)}{1-6+7-6+1}\\
&= 2.
\end{aligned}
$$

Solution 2 by Brian D. Beasley, Clinton, SC

Since $x^4+6x^3+7x^2+6x+1=(x^2+x+1)(x^2+5x+1)=0$, we calculate the four roots and assign the values $a=(-1+i\sqrt{3})/2$, $b=(-1-i\sqrt{3})/2$, $c=(-5+\sqrt{21})/2$, and $d=(-5-\sqrt{21})/2$. This yields: